Invader Strategies in the War of Attrition with Private Information*

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Abstract

Second price allpay auctions (wars of attritions) have an evolutionarily stable equilibrium in pure strategies if valuations are private information. I show that for any level of uncertainty there exists a pure deviation strategy close to the equilibrium strategy such that for some valuations the equilibrium strategy has a selective disadvantage against the deviation if the population mainly plays the deviation strategy. There is no deviation strategy with this destabilizing property for all valuations if the distribution of valuations has a monotonic hazard rate. I argue that in the Bayesian game studied here, a mass deviation can be caused by the entry of a small group of agents. Numeric calculations indicate that the closer the deviation strategy to the equilibrium strategy, the less valuations are destabilizing. I show that the equilibrium strategy does not satisfy continuous stability.

Keywords: Continuous Strategies, Evolutionary Stability, War of Attrition, Strict Equilibrium, Neighborhood Invader Strategy, Continuous Stability, Evolutionary Robustness.

JEL classification: C72, C73, D44

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Extended Abstract

This paper analyzes the stability of equilibrium behavior in second price allpay auctions of incomplete information. An allpay auction is a contest in which each contestant exerts efforts that are foregone regardless of winning the prize. In a second price allpay auction, the winner pays the second highest bid and all other contestants pay their own bid. This contest is also known as the 'War of Attrition'. Beside the biological interpretation, electoral first-pass-the-post campaigns, lobbying, academic research, public invitations to tender, and irreducible investments with conditional stochastic yields are all examples in which monetary or non-monetary spendings are sunk before the final allocation of the prize is fixed. A strategy is a mapping from a continuum of valuations into the non-negative reals. For such strategies the literature does not agree on the notion of stability. If the valuations are equal and fixed for each participant, the unique equilibrium has full support and satisfies all stability conditions defined for games with continuous strategies such as neighborhood invader strategy, continuously stable strategy, and in particular the strongest: evolutionary robustness. This is in line with Maynard Smith's (1974) finding that the equilibrium is evolutionarily stable, a stability concept defined for games with finitely many strategies. In the game in which valuations are only privately known, I prove the existence of a strategy which is destabilizing for an open set of valuations for any continuous distribution of valuations. This contrasts the finding by Bishop, Cannings and Maynard Smith (1978) that the equilibrium is evolutionarily stable. A biological interpretation of private valuations is the intensity of hunger which cannot be observed directly by the contestant, economic interpretations refer to non-monetary private preferences for the resource or unobservable cost intensities for effort exertion. I show that there is no strategy with such a destabilizing property for all valuations. The stability concepts defined for games with continuous strategy sets require stability against mass deviations, that is if the whole population simultaneously and identically deviates. While in games of complete information such a deviation is of least plausibility, in the war of attrition with incomplete information a mass deviation can be triggered by a mutation of an arbitrary small fraction of the population that changes its valuations. This mutation changes the distribution of valuations within the population and hereby the equilibrium strategy. The initital mutation of the small fraction is as if the whole population simultaneously and identically deviates to a non-equilibrium strategy.

1 Introduction & Literature

In this paper I analyze the stability of equilibrium behavior in second price allpay auctions of incomplete information with two contestants. An allpay auction is a contest in which each contestant exerts efforts that are foregone regardless of winning the prize. In a second price allpay auction, the winner pays the second highest bid and all other contestants pay their own bid. This contest is also known as the 'War of Attrition' which was introduced by Maynard Smith (1974). Beside the biological interpretation, an allpay auction is a situation to which social agents are exposed in daily routine: a successful job market candidate needs to be better qualified than the second best candidate, a sprinter needs to poke his or her nose a fraction of a second over the finish line before the second fastest athlete. Electoral first-pass-the-post campaigns, lobbying, academic research, public invations to tender, and irreducible investments with conditional stochastic yields are all examples in which monetary or non-monetary spendings are sunk before the final allocation of the prize is fixed. These situations also share the property that the absolute value of the bid is irrelevant - what matters is relative bid intensities. I study contests in which the valuation for the prize is private information. Situations in which the prize is equally valuable for all contestants but the cost of exerting efforts differ are every bit as plausible as the setting chosen here and can be seen as equivalent after a transformation of payoffs.

Maynard Smith (1974)'s 'War of Attrition' and related allpay auctions have been shown to be the limit of other, more general models as in Abreu & Gul (2000) who develop a model of reputation based bargaining or Lang et al. (2010) who analyze stochastic (Poisson) contests and Che & Gale (2000) who analyze rent seeking games. Bishop, Cannings & Maynard Smith (1978) characterize the ESS for the case of incomplete information. Milgrom & Weber (1985) show that as uncertainty approaches zero, the distribution of the (pure) bids converges to the mixed strategy distribution of Maynard Smith (1974). The War of Attrition with incomplete information has also been studied by Nalebuff & Riley (1985) and Ponsati & Sákovics (1995).

It is understood that the War of Attrition, or other allpay auctions can be found in many economic applications, such as the IO models 'The Generalized War of Attrition' in Bulow & Klemperer (1999) or Konrad (2006).

Beside Maynard Smith (1974), allpay auctions with complete information have been studied by Tullock (1980), Baye, Kovenock & De Vries (1996), Siegel (2009), and Moldovanu & Sela (2001, 2006)

Rose (1978) studies the evolutionary stability of allpay first price auctions (Scotch Auctions), the stability of first price auctions in which only the winner pays has been studied by Hon-Snir, Monderer & Sela (1998) and Louge & Riedel (2010). The War of Attrition in finite populations has been studied by Riley (1980), allpay auctions (Tullock-contest) have been shown to exhibit non-Nash behavior for finite populations by Leininger (2009). Damianov, Oechssler & Becker (2010) investigate whether a uniform or a discriminatory price auction is better for the seller in an experiment.

Bishop, Cannings & Maynard Smith (1978) use the concept of evolutionary stability in a game with continuous strategies. For such games it has been proposed to use other concepts as neighborhood invader strategy (NIS, Apaloo (1997, 2006)), continuously stable strategy (CSS, Eshel (1983)), evolutionary robustness ($\mathcal{ER}$, Oechssler & Riedel (2002), and asymmetric CSS and NIS (Cressman (2010)), because it has been shown that evolutionary stability is not sufficient for dynamic stability if strategies are continuous. Already Bishop & Cannings (1978) show convergence to the ESS in their 'Generalized War of Attrition' only for finite strategy sets. To stress that the critique of the use of ESS is long known I quote Hofbauer, Schuster & Sigmund (1979), p.611:

> "(...) [I]t could be that under certain circumstances it would be more appropriate to study asymptotically stable equilibria of (l), rather than ESS."

The referred to equation (1) is the replicator dynamic.

The effect of discretization of a continuous game is the subject of Alós-Ferrer (2006). Also Boudreau (2010) studies allpay auctions with discrete action spaces.

Krishna & Morgan (1997) develop a model in which allpay auctions raise more expected revenue than other sealed-bid auction forms. Leininger (2000) sees the allpay auction as a benchmark lottery and discusses the role of the revenue equivalence theorem in understanding the differences of auction types.

This paper adds to the literature that analyzes the dynamic stability of equilibrium strategies in auctions. In the current setting, a strategy is a mapping from a continuum of types (valuations) into the non-negative reals. For such strategies the literature does not agree on the notion of stability. I prove the existence of an invader strategy which is destabilizing for an open set of valuations for any continuous distribution of valuations. I show that there is no strategy with such a property for all valuations. I show that the equilibrium strategy is not continuously stable (Eshel (1983)). I hereby claim that there is no good argument for dynamic stability of the equilibrium strategy in the war of attrition with private valuations. Section 2 presents the static model and its equilibrium, section 3 discusses the use of the stability concept. Sections 4 and 5 collect the analytic respective the numeric results and section 6 concludes.

2 The Static Model

Let there be two contestants, each having a valuation in the set $\mathcal{V} \subset \mathbb{R}_+$, where $\mathcal{V}$ is an interval containing the valuation $\bar{v}$. The valuations are distributed according to a cdf F with continuous positive density f. Let $\mathcal{B} = \mathbb{R}_+$ be the set of bids that a contestant can choose from. A pure bid-strategy is a mapping $\beta : \mathcal{V} \to \mathcal{B}$ that assigns for each valuation $v \in \mathcal{V}$ a bid $\beta(v) \in \mathcal{B}$. If one contestant uses strategy β, the other contestant with bid b and valuation v expects to receive payoffs

$$\pi(b|v,\beta) = \begin{cases} \int_{\{w:\beta(w)<b\}} (v - \beta(w)) f(w) dw & \text{gets prize and pays opponent's bid.} \\ + \int_{\{w:\beta(w)=b\}} (\frac{v}{2} - \beta(w)) f(w) dw & \text{gets half of the prize and pays opponent's bid.} \\ - \int_{\{w:\beta(w)>b\}} b f(w) dw & \text{does not get the prize and pays own bid.} \end{cases}$$

If $F(\{w : \beta(w) = b\}) = 0$ for all $b \in \mathcal{B}$, the payoffs can be expressed as

$$\pi(b|v,\beta) = \int_{\{w:\beta(w)<b\}} (v + b - \beta(w)) f(w) dw - b \ .$$

2.1 Equilibrium

Bishop, Cannings & Maynard Smith (1978) show that the unique Bayesian Nash equilibrium consists of the strategy

$$\beta(v) = \int_0^v \frac{w f(w)}{1 - F(w)} dw \ .$$

If F is the uniform distribution on $[0, 1]$, then

$$\beta(v) = -\ln(1 - v) - v$$

Figure (2.1) below depicts the contour curves of $\pi(b|v, \beta)$ and the equilibrium strategy β (dashed line) if valuations are uniform on $[0, 1]$.

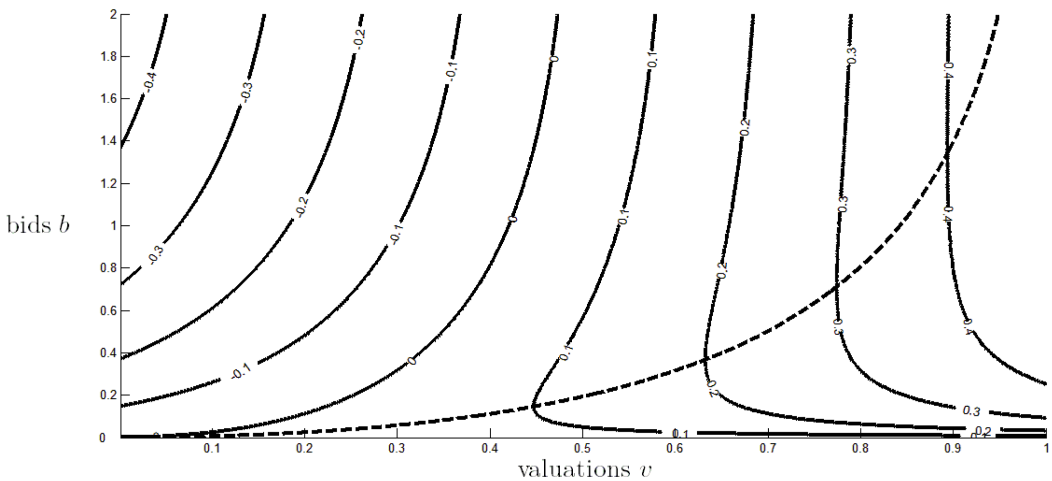

Figure 1: level curves and equilibrium strategy for uniform valuations

If the valuation is fixed at $\bar{v}$, $F(\{w : w = \bar{v}\}) = 1$ Maynard Smith (1974) shows that the unique symmetric equilibrium[1] consists of the mixed strategy $\sigma(b) = \frac{1}{\bar{v}} e^{-b/\bar{v}}$. Maynard Smith (1974) shows that the mixed strategy is an ESS in the game of complete information and Bishop, Cannings & Maynard Smith (1978) show evolutionary stability of the pure strategy equilibrium in the game of incomplete information.

[1]There is an asymmetric equilibrium (b, d) with $b \geq \bar{v}$ and $d = 0$.

Milgrom & Weber (1985) argue that if F is uniformly concentrated on a neighborhood $(\bar{v}-\epsilon, \bar{v}+\epsilon)$, then in equilibrium (β, β) the distribution of bids converges to the distribution induced by $\sigma(b)$ if $\epsilon \to 0$.

3 Dynamic Stability

To analyze dynamic stability, we have the following interpretation of the model: suppose that there is an infinite population of contestants each having a fixed valuation such that the distribution of valuations matches F. To play the contest, two agents are independently and uniformly matched. For each agent a strategy is an element of $\mathcal{B}$ rather than a mapping $\beta : \mathcal{V} \to \mathcal{B}$. A stability condition needs to hold for each single valuation v. Bishop, Cannings & Maynard Smith (1978) show that the condition for evolutionary stability does hold for each valuation. As several authors pointed out, the standard notion of ESS is not sufficient for dynamic stability in games of infinite strategies. For such games CSS, NIS, and $\mathcal{ER}$ have been proposed.

Any of these concepts require stability against mass deviations. A mass deviation describes a situation in which each agent of the population simultaneously deviates to an identical strategy. This seems to be a very odd and implausible event as 'trembles' of 'mutations' usually are seen as independent events. Why can the whole population independently and undirectedly 'mutate' to the same deviation strategy? In the Bayesian game considered here, I show below that there is a correlation device that gives a plausible interpretation for mass deviations and that this mass deviations is triggered by an arbitrarily small subgroup of agents.

3.1 discrete vs continuous strategy sets

Consider a game with two pure strategies "0" and "1". Suppose the current state is that all agents play "0". Let us have a brief view on two distinct deviations:

Deviation A: a small fraction ϵ of agents deviate to the strategy "1".
Deviation B: all agents deviate to the mixed strategy $(1-\epsilon) \cdot$ "0" $+ \epsilon \cdot$ "1".

Let σ_ϵ^i be the pure strategy played by a randomly chosen agent after the deviation $i \in \{A, B\}$. Then $\text{Prob}(\sigma_\epsilon^A =$ "0"$) = 1-\epsilon = \text{Prob}(\sigma_\epsilon^B =$ "0"$)$ and

Prob(σ_ϵ^A = "1") = ϵ = Prob(σ_ϵ^B = "1"). The type of deviation is irrelevant for the payoffs of an individual.

Consider now the continuous strategy set $S = [0,1]$ with the current state δ_0.[2] Deviation A would correspond to the distribution $\sigma_\epsilon^A = (1-\epsilon)\cdot\delta_0 + \epsilon\cdot\delta_1$, a small fraction of agents deviates to strategy 1 and deviation B would be $\sigma_\epsilon^B = \delta_\epsilon$, the whole population deviates to strategy ϵ close to strategy zero. How close are σ_ϵ^A and σ_ϵ^B to δ_0? The answer hereto depends on the used measure of distance:

For two functions $f, g : S \to \mathbb{R}$ define

$$\epsilon_x^A = \min\{\epsilon \geq 0 : f(x) \leq g(x) + \epsilon \text{ and } f(x) + \epsilon \geq g(x)\}$$

and

$$\epsilon_x^B = \min\left\{\epsilon \geq 0 : f(x) \leq \int_{x-\epsilon}^{x+\epsilon} g(x)dx \text{ and } \int_{x-\epsilon}^{x+\epsilon} f(x)dx \geq g(x)\right\} .$$

Define $d_i(f,g) = \max\{\epsilon_x^i : x \in S\}$. Then

$$d_A(\delta_0, \sigma_\epsilon^A) = \epsilon \text{ and } d_A(\delta_0, \sigma_\epsilon^B) = 1$$

and

$$d_B(\delta_0, \sigma_\epsilon^A) = 1 \text{ and } d_B(\delta_0, \sigma_\epsilon^B) = \epsilon.$$

Depending on the choice of the measure of distance, one kind of deviation is close to the equilibrium strategy while the other is not. If the standard definition of the ESS is used for continuous strategies, then an ESS is stable against deviations that are close in the sense of d_A. The concepts CSS and NIS use d_B. A strategy is $\mathcal{ER}$ if it is stable against deviations that are close in the sense of either d_A or d_B, hence $\min\{d_A, d_B\}$. The metric for $\mathcal{ER}$ uses

$$\epsilon_x = \min\left\{\epsilon \geq 0 : f(x) \leq \int_{x-\epsilon}^{x+\epsilon} g(x)dx + \epsilon \text{ and } \int_{x-\epsilon}^{x+\epsilon} f(x)dx + \epsilon \geq g(x)\right\} ,$$

$$d(f,g) = \max\{\epsilon_x : x \in S\}$$

and is a simplified version of the Prohorov metric for the special case if $S \subset \mathbb{R}_+$. Note that $d_A(\cdot)$ and $d_B(\cdot)$ coincide on finite strategy sets.

[2] δ_x is the dirac measure on strategy x, the pure state in which all agents play x.

3.2 A justification for B-deviations

If F is uniform on $[0,1]$, the equilibrium strategy is $\beta(v) = -\ln(1-v) - v$ for all v. Imagine that f and F change slightly to

$$f_a(v) = \begin{cases} 1-a+4av & \text{if } v < \frac{1}{2} \\ 1+3a-4av & \text{if } v \geq \frac{1}{2} \end{cases} \text{ and } F_a(v) = \begin{cases} (1-a)v + 2av^2 & \text{if } v < \frac{1}{2} \\ (1+3a)v - a - 2av^2 & \text{if } v \geq \frac{1}{2} \end{cases}$$

for $a \in (0,1]$. If $a = 1$, then f_a is the density of the sum of two variables that are uniform on $[0, \frac{1}{2}]$. The equilibrium strategy changes to

$$\beta_a(v) = \begin{cases} \int_0^v \frac{w(1-a+4aw)}{1-(1-a)w-2aw^2} dw & \text{if } v < \frac{1}{2} \\ \int_0^{\frac{1}{2}} \frac{w(1-a+4aw)}{1-(1-a)w-2aw^2} dw + \int_{\frac{1}{2}}^v \frac{w(1+3a-4aw)}{1-(1+3a)w+a+2aw^2} dw & \text{if } v \geq \frac{1}{2} \end{cases}$$

If $a = 1$ then

$$\beta_1(v) = \begin{cases} \int_0^v \frac{4w^2}{1-2w^2} dw & \text{if } v < \frac{1}{2} \\ \int_0^{\frac{1}{2}} \frac{4w^2}{1-2w^2} dw + \int_{\frac{1}{2}}^v \frac{4w}{1-w} dw & \text{if } v \geq \frac{1}{2} \end{cases}.$$

A sudden change from f_a to f can then be seen as a B-deviation: given f

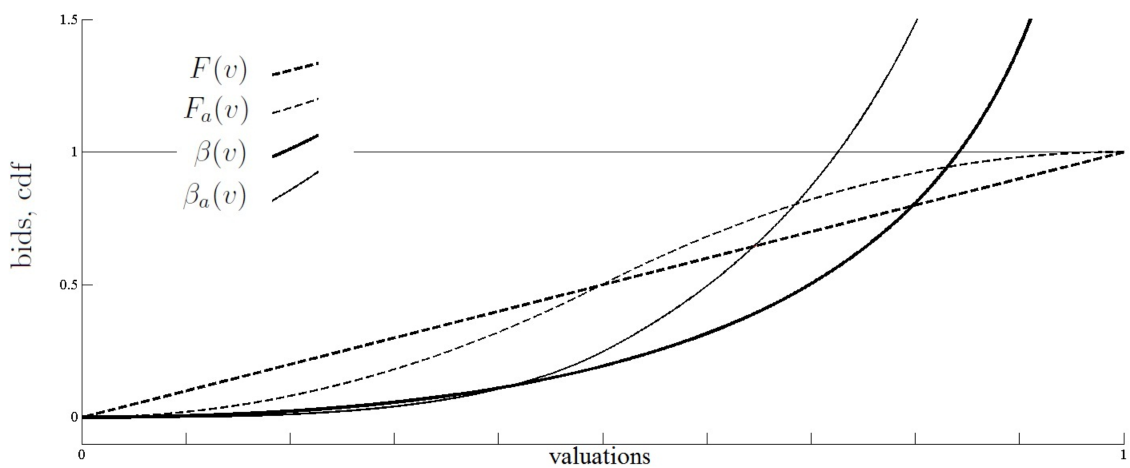

Figure 2: Equilibrium strategies for distributions F and F_a with $a = 1$.

the equilibrium strategy is β, but the population still plays the strategy β_a.

More generally, consider a situation in which the valuations are distributed according to a distribution function G with density g and the population plays equilibrium strategy γ with $\gamma(v) = \int_0^v \frac{wg(w)}{1-G(w)}dw$ for all $v \in \mathcal{V}$. If a small group of agents with fixed valuations enters the population, then the distribution of valuations changes slightly from G to F (and g to f). This invasion by a small group is *as if* the whole population simultaneously and identically deviates to strategy γ which is distinct and close to the equilibrium strategy β with $\beta(v) = \int_0^v \frac{wf(w)}{1-F(w)}dw$ for all $v \in \mathcal{V}$.

In the sections below I investigate whether the conditions are met such that agents of any type v change their strategy from $\gamma(v)$ to $\beta(v)$ in payoff monotonic dynamics.

3.3 Definitions

For symmetric games with finite sets of strategies an evolutionarily stable strategy is defined as a strategy that cannot be invaded by any similar strategy:

Definition 1 (ESS) *A strategy x is an evolutionarily stable strategy if*
$\pi(x,x) \geq \pi(y,x)$ *for any strategy y and if*
$\pi(x,x) = \pi(y,x)$ *for some y, then* $\pi(x,y) > \pi(y,y)$.

The logic behind the definition is that whenever a homogenous ESS population is invaded by a small portion of mutants, the ESS agents have a selective advantage against the mixture in which the population mainly plays the evolutionarily stable strategy and a small fraction mutates. Still, definition 1 technically allows for two interpretations of mutations. The first, and usually preferred interpretation is that mutations happen independently and undirected. This means that only a very small fraction of agents mutates and that the mutant strategy can be any strategy of the set of strategies. The second interpretation, merely from a correlated shocks perspective, is that almost all agents change their strategy, but the new strategy must be very similar to the original one.

Hofbauer, Schuster & Sigmund (1979) show that a strategy is ESS if and only if it satisfies $\pi(x,y) > \pi(y,y)$ for all $y \neq x$ if the game is finite. Weibull (1995) coins this condition

Definition 2 (local superiority, Weibull 1995) *x is locally superior if it has a neighborhood U such that $\pi(x,y) > \pi(y,y)$ for all $y \neq x$ in U.*

It is exactly this condition which was used by Hofbauer, Schuster & Sigmund (1979) to construct a Lyapunov function to show asymptotic stability. It will reappear in the definitions of stability that follow.

The following definition for neighborhood invader strategy says that if the population mainly uses a strategy y that is distinct to the neighborhood invader strategy x, x has a selective advantage over y.

Definition 3 (NIS, Apaloo 2006) *A strategy x is a neighborhood invader strategy if for any y: $\pi(x,y) \geq \pi(y,y)$ and if $\pi(x,y) = \pi(y,y)$ then $\pi(x,x) > \pi(y,x)$.*

Strategy x is a local NIS (Apaloo 1997), if the definition above holds for all y close to x. If NIS x is considered to be robust against mutations, the interpretation of stability for this concept is that the population mainly mutates to a strategy y which is close but selection still favors x.

Eshel (1983) also considers deviations of type B given some ESS x when asking

> "If a large enough majority of the population prefers a strategy y which is sufficiently close to x (...), will it be advantageous for each individual in this population to choose a strategy closer to, rather than further apart from x?"

Definition 4 (CSS, Eshel 1983) *An ESS x is continuously stable if there is a value $\epsilon > 0$ such that for any strategy y $\epsilon-$close to x there is some $\delta > 0$ such that for any strategy u at a δ-distance to y it holds that $\pi(u,y) > \pi(y,y)$ if and only if $|u - x| < |y - x|$.*

Eshel (1983) also offers a necessary and a sufficient condition for continuous stability which involve the second derivative of the payoff function with respect to the strategy of the opponent. In the game studied here, the strategy of the opponent is the population strategy which is a function mapping valuations to bids and the necessary and sufficient conditions cannot be applied.

The next definition is perhaps the strongest notion of dynamic stability in non-cooperative games as it it requires robustness against the largest set of deviations.

Definition 5 ($\mathcal{ER}$, Oechssler & Riedel 2002) *A strategy x is $\mathcal{ER}$ if $\pi(x,y) > \pi(y,y)$ for all $y \neq x$ that are ϵ-close to x in the Prohorov metric.*

The definition is originally stated for mixed strategies, for the purpose of this paper it suffices to give the defintion for pure strategies. Note that NIS $\not\Leftrightarrow \mathcal{ER}$ but that the local version of NIS is necessary for $\mathcal{ER}$.

In the next section I argue that for some valuations the equilibrium strategy β neither is a locally superior or locally NIS nor CSS and hereby that the conditions for $\mathcal{ER}$ are not met.

Maynard Smith (1974) shows for the mixed equilibrium $\sigma(b) = \frac{1}{\bar{v}}e^{-b/\bar{v}}$ that $\int_0^\infty \sigma(b)\pi(b,\delta_m)db > \pi(m,\delta_m)$ for all pure strategies $m \in \mathbb{R}_+$, hence $\pi(\sigma,\delta_m) > \pi(m,\delta_m)$ for all $m \in \mathbb{R}_+$. Therefore we may conclude for the war of attrition without private information that the mixed equilibrium is evolutionarily robust.

Corollary 1 *The fully mixed equilibrium in the war of attrition with complete information is evolutionarily robust.*

4 Propositions

While the equilibrium in the War of Attrition with complete information is mixed, the War of Attrition with incomplete information has an equilibrium in pure strategies. The first theorem notes that this equilibrium is strict.

Theorem 1 *The equilibrium given by $\beta(v) = \int_0^v \frac{wf(w)}{1-F(w)}dw$ is an equilibrium with unique best replies.*

PROOF : Let ϕ be the inverse of β, that is $\phi(\beta(v)) = v$ for all v. β is C^1 and strictly increasing hence ϕ exists and is also C^1 and strictly increasing. It is clear that $\phi'(\beta(v)) = \frac{1}{\beta'(v)}$ and $\phi''(\beta(v)) = -\frac{\beta''(v)}{\beta'(v)^3}$, therefore $\phi'(\beta(v)) =$

$\frac{1-F(v)}{vf(v)}$ and $\phi''(\beta(v)) = -\frac{(f(v)+vf'(v))(1-F(v))^2+vf(v)^2(1-F(v))}{(vf(v))^3}$.

$$\begin{aligned}
\pi(b|v,\beta) &= \int_0^{\phi(b)} (v+b-\beta(w))f(w)dw - b \\
\frac{\partial \pi(b|v,\beta)}{\partial b} &= vf(\phi(b))\phi'(b) + F(\phi(b)) - 1 \\
\frac{\partial^2 \pi(b|v,\beta)}{(\partial b)^2} &= vf'(\phi(b))(\phi'(b))^2 + vf(\phi(b))\phi''(b) + f(\phi(b))\phi'(b) \\
\Big|_{b=\beta(v)} &= -\frac{f(v)}{(\beta'(v))^2} < 0
\end{aligned}$$

□

Theorem 1 implies that β is evolutionarily stable. Note that Bishop et al. (1978) have used a different method to prove that β is an ESS. They analyzed a finite partition of the set of valuations $\mathcal{V}$ and interpreted their results for this partition becoming infinitesimal fine. Theorem 1 offers an important insight: as the first condition of ESS, namely $\pi(b,\beta) > \pi(\tilde{b},\beta)\ \forall \tilde{b} \neq b$ is satisfied for all b, the second condition does not need to be checked. In what follows I explore whether there exists a strategy that violates local superiority, the second condition of evolutionary stability.

Consider a strategy γ that is close but distinct to the equilibrium strategy β in the sense that there are small $\epsilon_1 > \epsilon_2 > 0$ such that $\epsilon_2 < |\gamma(v) - \beta(v)| < \epsilon_1$ for all $v \in \mathcal{V} : v > 0$. Suppose the population plays γ. Claim 1 shows that β fares weakly better against γ than γ against itself for any valuation $v > 0$ if the distribution of valuations satisfies a monotonicity assumption.

Definition 6 (MHR Barlow, Marshall & Proschan (1963)) *F satisfies the monotone hazard rate property if $\frac{vf(v)}{F(v)}$ is increasing in v for all valuations in $\mathcal{V}$.*

The Exponential-, Binomial-, Poisson-, Normal-, and uniform distributions all satisfy MHR. Note further that any cdf must satisfy MHR for at least some valuations v.

Claim 1 *If F satisfies MHR and if strictly increasing $\gamma(v)$ is close but distinct to $\beta(v)$ for all positive v, then $\pi(\gamma(v)|v,\gamma) < \pi(\beta(v)|v,\gamma)$ for all positive v.*

PROOF :

$$\begin{aligned}
\pi(b|v,\gamma) &= \int_{\{w:\gamma(w)<b\}} (v+b-\gamma(w))f(w)dw - b \\
\left.\frac{\partial \pi(b|v,\gamma)}{\partial b}\right|_{b=\beta(v)} &= \frac{vf(\{w:\gamma(w)=\beta(v)\})}{\frac{\partial \beta(v)}{\partial v}} - (1-F(\{w:\gamma(w)<\beta(v)\})) \overset{(>)}{<} 0 \\
&\Leftrightarrow \frac{vf(\{w:\gamma(w)=\beta(v)\})}{1-F(\{w:\gamma(w)<\beta(v)\})} \overset{(>)}{<} \frac{\partial \beta(v)}{\partial v} = \frac{vf(v)}{1-F(v)}
\end{aligned}$$

If $\gamma(v) < \beta(v)$, then $\{w : \gamma(w) = \beta(v)\}$ contains valuations that are larger than v and the derivative is positive. If $\gamma(v) > \beta(v)$, the derivative is negative. □

The claim has a weak foundation as the second derivative is not considered. Unfortunately, the sign of the second derivative is ambivalent,

$$\frac{\partial^2 \pi(b|v,\gamma)}{(\partial b)^2} = vf'(\gamma^{-1}(b))(\gamma^{-1\prime}(b))^2 + f(\gamma^{-1}(b))\left(v\gamma^{-1\prime\prime}(b) + \gamma^{-1\prime}(b)\right)$$

as there is no restriction for $\gamma'(v)$ and hence $\gamma^{-1\prime}(b)$ for $\gamma(v)$ close but distinct to $\beta(v)$.

The following theorem seems to test claim 1 given above. Consider a strategy γ that is defined given equilibrium strategy $\beta(v) = \int_0^v \frac{wf(w)}{1-F(w)}dw$. The following theorem proposes a candidate strategy γ such that in a population of agents playing γ, the strategy β would have a selective disadvantage in any dynamic that is based on relative payoffs.

Theorem 2 *If strictly increasing $\gamma : \mathcal{V} \to \mathbb{R}_+$ with $\gamma(v) > \beta(v)$ for all $v > 0$ solves*

$$\gamma(v) = \beta(v) + \lambda v \frac{F(\{w : \beta(v) < \gamma(w) < \gamma(v)\})}{F(\{w : \beta(v) < \gamma(w)\})}$$

for $\lambda \in (0,1)$, then $\pi(\gamma(v)|v,\gamma) > \pi(\beta(v)|v,\gamma)$ for all $v > 0$.

Proof :

$$
\begin{aligned}
& \pi(\gamma(v)|v,\gamma) - \pi(\beta(v)|v,\gamma) \\
= & \int_{\{w:\beta(v)<\gamma(w)<\gamma(v)\}} (v+\gamma(v)-\gamma(w))f(w)dw - (\gamma(v)-\beta(v))F\left(\{w:\beta(v)<\gamma(w)\}\right) \\
> & \; v\int_{\{w:\beta(v)<\gamma(w)<\gamma(v)\}} f(w)dw - (\gamma(v)-\beta(v))F\left(\{w:\beta(v)<\gamma(w)\}\right) \\
= & \; v(1-\lambda)F\left(\{w:\beta(v)<\gamma(w)\}\right) > 0 \;\forall\lambda<1
\end{aligned}
$$

□

Note the ostensible strength of the theorem with respect to the interpretation of the model that is used. As it holds for all valuations, $\int_{\mathcal{V}} \pi(\gamma(v)|v,\gamma)f(v)dv > \int_{\mathcal{V}} \pi(\beta(v)|v,\gamma)f(v)dv$, hence the conclusion also holds *ex ante* in a two player game without populations. This would indeed contradict claim 1.

The next theorem reveals the true strength of theorem 2 by stating that candidate γ must be equal to the equilibrium strategy β.

Theorem 3 *If*

$$\gamma(v) = \beta(v) + \lambda v \frac{F\left(\{w:\beta(v)<\gamma(w)<\gamma(v)\}\right)}{F\left(\{w:\beta(v)<\gamma(w)\}\right)} \;,$$

then $\gamma(v) = \beta(v)$.

Proof : Fix v and define the righthandside as function $\xi(\gamma_v)$. As ξ is continuous and

$$\frac{\partial\xi(\gamma_v)}{\partial\gamma_v} = -\lambda\frac{f(\{w:\beta(v)=\gamma(w)\})(1-F(v))}{F(\{w:\beta(v)<\gamma(w)\})^2} < 0 \;,$$

there can be at most one fixed point. Obviously, $\xi(\beta(v)) = \beta(v)$. Hence $\gamma(v) = \beta(v)$ and claim 2 is empty. □

Let us now consider a strategy γ that intersects the equilibrium strategy once at valuation $\bar{v}$. It is clear that such a function exists and can be arbitrary close to β.[3]

[3] For example, $\gamma(v) = \beta(v)^\theta$ for all $v < \bar{v}$ and $\gamma(v) = \beta(v)$ for all $v > \bar{v}$ and $\bar{v}$: $\beta(\bar{v}) = 1$.

Theorem 4 *The equilibrium strategy β is not a neighborhood invader strategy.*

PROOF : Let $\gamma : \mathcal{V} \to \mathbb{R}_+$ be continuous, have a positive derivative for all v and let γ intersect β at $\bar{v}$: $\gamma(\bar{v}) = \beta(\bar{v})$ with different slopes at $\bar{v}$.

$$\begin{aligned}
\pi(\beta(v)|v,\gamma) &= \int_{\{w:\gamma(w)<\beta(v)\}} (v+\beta(v)-\gamma(w))f(w)dw - \beta(v) \\
\pi(\gamma(v)|v,\gamma) &= \int_{\{w:\gamma(w)<\gamma(v)\}} (v+\gamma(v)-\gamma(w))f(w)dw - \gamma(v) \\
\frac{\partial\pi(\beta(v)|v,\gamma)}{\partial v} &= \left(1+\frac{\partial\beta(v)}{\partial v}\right)\cdot F\left(\{w:\gamma(w)<\beta(v)\}\right) \\
&+ \quad v\cdot f(\{w:\gamma(w)=\beta(v)\})\frac{\frac{\partial\beta(v)}{\partial v}}{\frac{\partial\gamma(v)}{\partial v}} - \frac{\partial\beta(v)}{\partial v} \\
\frac{\partial\pi(\gamma(v)|v,\gamma)}{\partial v} &= \left(1+\frac{\partial\gamma(v)}{\partial v}\right)\cdot F(v) + v\cdot f(v) - \frac{\partial\gamma(v)}{\partial v}
\end{aligned}$$

$$\left.\frac{\partial\pi(\gamma(v)|v,\gamma)}{\partial v}\right|_{v=\bar{v}} - \left.\frac{\partial\pi(\beta(v)|v,\gamma)}{\partial v}\right|_{v=\bar{v}} = -\frac{1-F(\bar{v})}{\frac{\partial\gamma(\bar{v})}{\partial v}}\left(\frac{\partial\gamma(\bar{v})}{\partial v}-\frac{\partial\beta(\bar{v})}{\partial v}\right)^2 < 0$$

By continuity of $\pi(\cdot)$ in v, $\pi(\gamma,\gamma) > \pi(\beta,\gamma)$ for all $v \in (\bar{v}-\epsilon, \bar{v})$. □

Theorem 5 *The equilibrium strategy β is not continuously stable.*

PROOF : Consider a continuously differentiable and increasing strategy γ that intersects β at $\bar{v}$ and has $|\gamma(v)-\beta(v)| < \epsilon$ for all v and some positive ϵ. Then, for all $\tilde{v}$ smaller than and close to $\bar{v}$:

$$(\gamma(\tilde{v})-\beta(\tilde{v}))(\gamma'(\tilde{v})-\beta'(\tilde{v})) < 0\ .$$

$$\begin{aligned}
\pi(b|\tilde{v},\gamma) &= \int_0^{\gamma^{-1}(\tilde{v})} (v+b-\gamma(w))f(w)dw - b \\
\frac{\partial\pi(b|\tilde{v},\gamma)}{\partial b} &= \tilde{v}f(\gamma^{-1}(b))\frac{\partial\gamma^{-1}(b)}{\partial b} + F(\gamma^{-1}(b)) - 1 \\
\left.\frac{\partial\pi(b|\tilde{v},\gamma)}{\partial b}\right|_{b=\gamma(\tilde{v})} &= \frac{1-F(\tilde{v})}{\gamma'(\tilde{v})}(\beta'(\tilde{v})-\gamma'(\tilde{v}))
\end{aligned}$$

Define $\tilde{\gamma}_+$ and $\tilde{\gamma}_-$ such that $\gamma_+(v) = \gamma_-(v) = \gamma(v)\ \forall\ v \neq \tilde{v}$, $\tilde{\gamma}(\tilde{v}) = \gamma(\tilde{v}) + \delta$, and $\tilde{\gamma}_-(v) = \gamma(\tilde{v}) - \delta$.

If $\gamma(\tilde{v}) \overset{(<)}{>} \beta(\tilde{v})$ we have $\pi(\tilde{\gamma}_+(\tilde{v})|\tilde{v},\gamma) \overset{(<)}{>} \pi(\gamma(\tilde{v})|\tilde{v},\gamma)$ and $\pi(\tilde{\gamma}_-(\tilde{v})|\tilde{v},\gamma) \overset{(>)}{<} \pi(\gamma(\tilde{v})|\tilde{v},\gamma)$, because $\pi(b|v,\gamma)$ is continuous in b.

γ is ϵ-close to β, $\tilde{\gamma}$ is δ-close to γ for arbitrary small δ and $\pi(\tilde{\gamma}(\tilde{v})|\tilde{v},\gamma) < \pi(\gamma(\tilde{v})|\tilde{v},\gamma)$ whenever $\tilde{\gamma}$ is closer to β and vice versa. Hence β is not CSS. □

Proposition 1 *Denote by $\hat{v}$ the greatest intersection of γ and β that is smaller than $\bar{v}$. There is a $\tilde{v} \in (\hat{v}, \bar{v})$ such that $\pi(\gamma(v)|v,\gamma) > \pi(\beta(v)|v,\gamma)\ \forall\ v \in (\tilde{v}, \bar{v})$.*

PROOF : By theorem 4 $\pi(\beta(v)|v,\gamma) > \pi(\gamma(v)|v,\gamma)$ for all v close to and greater than $\hat{v}$. By continuity of π there must be an intermediate value $\tilde{v} < \bar{v}$ such that $\pi(\gamma(\tilde{v})|\tilde{v},\gamma) = \pi(\beta(\tilde{v})|\tilde{v},\gamma)$. As $\hat{v}$ is the greatest intersection of β and γ that is smaller than $\bar{v}$, $\tilde{v}$ is the unique intersection of $\pi(\gamma(v)|v,\gamma)$ and $\pi(\beta(v)|v,\gamma)$ between $\hat{v}$ and $\bar{v}$. □

As a consequence, we can find for any positive ϵ a strategy γ that is close to β in the sense of the Prohorov metric, $d(\delta_\gamma, \delta_\beta) < \epsilon$ such that $\tilde{v} < \bar{v}$ and $\pi(\gamma(v)|v,\gamma) > \pi(\beta(v)|v,\gamma)$ for all valuations $v \in (\tilde{v}, \bar{v})$.

5 Numeric Calculations

Consider a uniform distribution on $[0,1]$ and

$$\gamma(v) = \begin{cases} \beta(v)^\theta & \text{if } v < \bar{v} \\ \beta(v) & \text{if } v \geq \bar{v} \end{cases}.$$

With this parametrization there are two intersections of β and γ where the smaller intersection is at $\hat{v} = 0$. The figures (3) - (4) below depict the strategies and payoffs for $\theta = 10$ and $\theta = 1.1$: Numeric calculations reveal that for $\theta \to 1$ the value $\tilde{v}$ where $\pi(\beta(v)|v,\gamma)$ hits $\pi(\gamma(v)|v,\gamma)$ from below approaches $\bar{v}$.

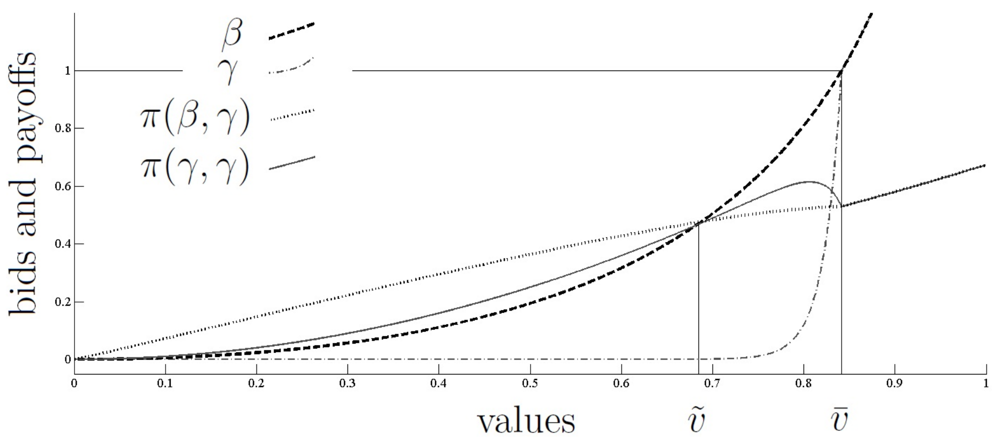

Figure 3: bids and payoffs for $\theta = 10$.

6 Conclusion

In this paper I analyze the dynamic stability of equilibria in second price allpay auctions with continuous bids and incomplete information on the valuation of the opponent. This is of particular interest as Bishop, Cannings & Maynard Smith (1978) show the existence of a unique ESS but Bishop & Cannings (1978) show convergence only for finite strategies. This paper aims at explaining the gap. In games with finite strategy sets an ESS cannot be invaded by neither independent and undirected deviations nor correlated and close deviations. Bishop, Cannings & Maynard Smith (1978) test their ESS only against independent deviations. I show that there exists a correlated mass deviation strategy that violates local superiority for an open set of valuations. The share of these valuations is constant and does not depend on the level of uncertainty. I give a plausible interpretation of mass deviations which is valid for all Bayesian games. I show that the equilibrium strategy does not satisfy continuous stability.

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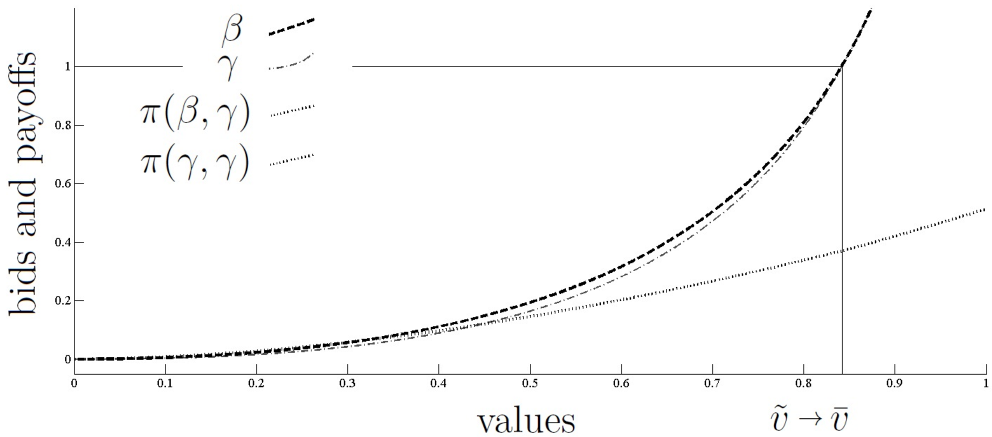

Figure 4: bids and payoffs for $\theta = 1.1$.

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